



4.2 Other norms for finite-dimensional vectors

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Introduction

- In this topic, we will
 - Define the 1-norm and the infinity-norm for finite-dimensional vectors $\mathbf{R}^n$ and $\mathbf{C}^n$
 - Observe they continue to satisfy the properties of a norm
 - Explain their application





Norms of vectors

- If $\mathbf{u} \in \mathbf{F}^n$, we will define

The 1-norm

$$\|\mathbf{u}\|_1 = \sum_{k=1}^n |u_k|$$

The ∞ -norm

$$\|\mathbf{u}\|_\infty = \max_{k=1,\dots,n} \{|u_k|\}$$





Norms of vectors

- Thus, if $\mathbf{u} = \begin{pmatrix} 0.3 \\ -0.5 \\ -1.3 \\ 0.2 \\ -0.7 \end{pmatrix}$, then

$$\begin{aligned}\|\mathbf{u}\|_2 &= \sqrt{0.09 + 0.25 + 1.69 + 0.04 + 0.49} \\ &= \sqrt{2.56} = 1.6\end{aligned}$$

$$\|\mathbf{u}\|_1 = 0.3 + 0.5 + 1.3 + 0.2 + 0.7 = 3$$

$$\|\mathbf{u}\|_\infty = 1.3$$





Norms of vectors

- Both of these also satisfy the three properties:

$$\begin{array}{ll} \|\mathbf{u}\|_1 \geq 0 & \text{and} \quad \|\mathbf{u}\|_1 = 0 \\ \|\mathbf{u}\|_\infty \geq 0 & \text{and} \quad \|\mathbf{u}\|_\infty = 0 \end{array} \quad \text{if and only if } \mathbf{u} = \mathbf{0}$$

$$\|\alpha \mathbf{u}\|_1 = |\alpha| \|\mathbf{u}\|_1$$

$$\|\alpha \mathbf{u}\|_\infty = |\alpha| \|\mathbf{u}\|_\infty$$

$$\|\mathbf{u} + \mathbf{v}\|_1 \leq \|\mathbf{u}\|_1 + \|\mathbf{v}\|_1$$

$$\|\mathbf{u} + \mathbf{v}\|_\infty \leq \|\mathbf{u}\|_\infty + \|\mathbf{v}\|_\infty$$

$$\|\mathbf{u}\|_1 = \sum_{k=1}^n |u_k|$$

$$\|\mathbf{u}\|_\infty = \max_{k=1, \dots, n} \{|u_k|\}$$





Norms of vectors

- Why would an engineer use a different norm?
 - Usually, engineers use the 2-norm
 - However, what if you're a tourist in New York
 - Is the distance 267 m or 382 m? That is, $\|\mathbf{u} - \mathbf{v}\|_2$ or $\|\mathbf{u} - \mathbf{v}\|_1$





Norms of vectors

- Why would an engineer use a different norm?
 - Consider moving the head of a 3-d printer
 - The time it takes to get from $\mathbf{u}$ to $\mathbf{v}$ depends on $\|\mathbf{u} - \mathbf{v}\|_\infty$



Josef Prusa





Norms of vectors

- In this course,
 - we will almost exclusively use the 2-norm
 - However, there are scenarios where engineers will use other, and more suitable, norms





Summary

- Following this topic, you now
 - Are aware of the 1- and infinity-norms
 - Understand they have properties similar to the 2-norm
 - Are aware that there are applications of other norms





References

- [1] [https://en.wikipedia.org/wiki/Norm_\(mathematics\)](https://en.wikipedia.org/wiki/Norm_(mathematics))





Acknowledgments

None so far.





Colophon

These slides were prepared using the Cambria typeface. Mathematical equations use Times New Roman, and source code is presented using Consolas.

The photographs of flowers and a monarch butter appearing on the title slide and accenting the top of each other slide were taken at the Royal Botanical Gardens in October of 2017 by Douglas Wilhelm Harder. Please see

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for more information.





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